Hyers-Ulam stability of second order non-linear Ordinary and Partial differential equation

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Abstract : In this paper, we have proved the Hyers -Ulam (HU) stability of second order non-linear Ordinary differential equation $u_{xx}(x,t) = f(x,t,u(x,t),u_x(x,t))$ and the Hyers -Ulam (HU) stability of second order non-linear Partial differential equation $u_{xt}(x,t) = f(x,t,u(x,t),u_x(x,t))$.

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1 Introduction

In 1940, S. M. Ulam [17] gave a famous talk regarding the stabiliy problems for various funcional equations. In the talk, Ulam discussed a problem concerning the stability of group homomorphism. In 1941, Hyers [5] gave a partial solution to Ulam's problem. In 1998, Alsina and Ger [1] investigated the Hyers - Ulam stability of the differential equation $y' = y$. In 2002, Takahasi et al. [16] generalized the result for $y' = \lambda y$. In 2009, Jung [6] proved the Hyers - Ulam stability of linear partial differential equations of first order. Rassias [13], focused on the stability of linear mapping in Banach spaces. H. Rezaei [14] established the HU stability of lineae differential equation of n^{th} order. For more resullts on Hyers - Ulam stability and related topics refer to [3], [4] and [7] to [10].

In 2011, Gordji et al. [2] proved the Hyers - Ulam- Rassias (HUR) stability of nonlinear partial differential equations by using Banach contraction principle. In 2019, Sonalkar et al.

[15] proved the HUR stability of first and third order linear partial differential equations by using Laplace Transform. Generalised HUR stability of a hyperbolic partial differential equation were studied by N. Lungu et al. [11] .

In this paper, by using the results in [11], we proved the HU stability for the second order non-linear ordinary differential equation

$$u_{xx}(x,t) = f(x,t,u(x,t),u_x(x,t)), \;\; 0 \le x \le a, 0 \le t \le b, \tag{1.1}$$

where $a, b \in (0,\infty)$, $f \in C([0,a]\times[0,b]\times\mathbb{B}^2,\mathbb{B})$ and $(\mathbb{B}, ||.||)$ be a real or complex Banach space.

Also we proved the HU stability for the second order non-linear partial differential equation

$$u_{xt}(x,t) = f(x,t,u(x,t),u_x(x,t)) \;\; 0 \le x \le a, 0 \le t \le b. \tag{1.2}$$

2 Preliminaries

In this section we present some definitions and results.

Definition 2.1 : Equation (1.1) is HU stable if $\exists$ real constants $c_f^1, c_f^2 > 0$ such that for any $\epsilon > 0$ and for any solution $v(x,t)$ of the inequality

$$||v_{xx}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \le \epsilon, \quad x \in [0,a], t \in [0,b], \tag{2.1}$$

$\exists$ a solution $u(x,t)$ of (1.1) with $||v(x,t) - u(x,t)|| \le \epsilon\, c_f^1$ and
$||v_x(x,t) - u_x(x,t)|| \le \epsilon\, c_f^2, \;\; \forall\, x \in [0,a], \forall\, t \in [0,b]$.

Remark 2.2 : A function $v(x,t)$ is a solution to the inequality (2.1) iff $\exists$ a continuous function $g(x,t)$ which depends on $v(x,t)$ such that
i) $||g(x,t)|| \le \epsilon$,
ii) $\forall\, x \in [0,a], \forall\, t \in [0,b], v_{xx}(x,t) = f(x,t,v(x,t),v_x(x,t)) + g(x,t)$.

Definition 2.3 : Equation (1.2) is HU stable if $\exists$ real constants $c_f^3, c_f^4 > 0$ such that for any $\epsilon > 0$ and for any solution $v(x,t)$ of the inequality

$$||v_{xt}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \le \epsilon, \;\; x \in [0,a], \;\; t \in [0,b], \tag{2.2}$$

$\exists$ a solution $u(x,t)$ of (1.2) with
$||v(x,t) - u(x,t)|| \le \epsilon\, c_f^3$ and
$||v_x(x,t) - u_x(x,t)|| \le \epsilon\, c_f^4, \forall\, x \in [0,a], \forall\, t \in [0,b]$.

Remark 2.4 : A function $v(x,t)$ is a solution to the inequality (2.2) iff $\exists$ a continuous function $g(x,t)$ which depends on $v(x,t)$ such that
i) $||g(x,t)|| \le \epsilon$,
ii) $\forall\, x \in [0,a], \forall\, t \in [0,b], v_{xt}(x,t) = f(x,t,v(x,t),v_x(x,t)) + g(x,t)$.

In proving main results following Gronwall type Lemma is required.
Lemma 2.5 : (see [12]) One assumes that
i) $u, v, h \in C(\mathbb{R}_+^n, \mathbb{R}_+)$,
ii) for any $t \ge t_0$, $u(t) \le h(t) + \int_{t_0}^t v(s)u(s)ds$.
iii)$h(t)$ is positive and increasing.

Then $u(t) \leq h(t) \times exp\{\int_{t_0}^{t} v(r)dr\}$, for any $t \geq t_0$.

Theorem 2.6 : If $v(x,t)$ is a solution to the inequality (2.1) then (v, v_x) satisfies the following integral inequality system

$||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \leq \frac{\epsilon x^2}{2}$,

$||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \epsilon x$.

Proof. If $v(x,t)$ is a solution to the inequality

$||v_{xx}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \epsilon$.

Integrating w.r.t. x we get,

$\int_0^x ||v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds \leq \int_0^x \epsilon ds$.

Since

$|| \int_0^x \{v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x ||v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds$,

we get,

$\Rightarrow || \int_0^x \{v_{ss}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x \epsilon ds$.

$||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \epsilon x$.

Integrating w.r.t. x we get

$\int_0^x ||v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds||dy \leq \int_0^x \epsilon sds$.

Since $|| \int_0^x \{v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds\}dy||$
$\leq \int_0^x ||v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds||dy$,

we get,

$\Rightarrow || \int_0^x \{v_y(y,t) - v_y(0,t) - \int_0^y f(s,t,v(s,t),v_s(s,t))ds\}dy|| \leq \int_0^x \epsilon sds$.

$\Rightarrow ||v(x,t) - v(0,t) - \{v(0,t) - v(0,t)\} - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy||$
$\leq \frac{\epsilon x^2}{2}$.

$\Rightarrow ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \leq \frac{\epsilon x^2}{2}$. □

Theorem 2.7 : : If $v(x,t)$ is a solution to the inequality (2.2) then (v, v_x, v_t) satisfies the following integral inequalities

$||v(x,t) - v(x,0) - v(0,t) + v(0,0) - \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz|| \leq \epsilon xt$,

$||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \epsilon x$,

$||v_x(x,t) - v_x(x,0) - \int_0^t f(x,z,v(x,z),v_x(x,z))dz|| \leq \epsilon t$.

Proof : If $v(x,t)$ is a solution to the inequality

$$||v_{xt}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \epsilon, \tag{2.3}$$

Integrating w. r. t. x we get,

$\int_0^x ||v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds \leq \int_0^x \epsilon ds$.

Since

$|| \int_0^x \{v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x ||v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t))||ds$,

we get,

$\Rightarrow || \int_0^x \{v_{st}(s,t) - f(s,t,v(s,t),v_s(s,t))\}ds|| \leq \int_0^x \epsilon ds$.

$\Rightarrow ||v_t(x,t) - v_t(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \epsilon x$.

Integrating w. r. t. t we get,

$\int_0^t ||v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z))ds||dz \leq \int_0^t \epsilon xdz$.

Since
$|| \int_0^t \{v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z))ds\}dz||$
$\leq \int_0^t ||v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z))ds||dz,$
we get,

$\Rightarrow || \int_0^t \{v_z(x,z) - v_z(0,z) - \int_0^x f(s,z,v(s,z),v_s(s,z))ds\}dz|| \leq \int_0^t \epsilon x dz.$
$\Rightarrow ||v(x,t) - v(x,0) - v(0,t) + v(0,0) - \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz|| \leq \epsilon xt.$

Integrating equation (2.3) w. r. t. t we get,

$\int_0^t ||v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z))||dz \leq \int_0^t \epsilon dz.$
Since
$|| \int_0^t \{v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z))\}dz|| \leq \int_0^t ||v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z))||dz,$
we get,

$\Rightarrow || \int_0^t \{v_{xz}(x,z) - f(x,z,v(x,z),v_x(x,z))\}dz|| \leq \int_0^t \epsilon dz.$

$\Rightarrow ||v_x(x,t) - v_x(x,0) - \int_0^t f(x,z,v(x,z),v_x(x,z))dz|| \leq \epsilon t.$

3 Hyers-Ulam stability for the second order non-linear ordinary differential equation :

In this section we prove the HU stability of second order non-linear ordinary differential equation (1.1) .

Theorem 3.1: Assume that
i) $f \in C([0,a] \times [0,b] \times \mathbb{B}^2, \mathbb{B})$, where $a, b \in (0,\infty)$.
ii)There exists $L_f \in C^1([0,\infty) \times [0,\infty), \mathbb{R}_+)$ such that L_f is square inegrable and $||f(x,t,z_1,z_2) - f(x,t,t_1,t_2)|| \leq L_f \underset{i\in\{1,2\}}{max} \{||z_i - t_i||\}, \forall\, x \in [0,a], \forall\, t \in [0,b]$ and $z_1, z_2, t_1, t_2 \in \mathbb{B}$.
iii) $\phi : (0,\infty) \times (0,\infty) \to (0,\infty)$ is an increasing function.
Then (1.1) is Hyers - Ulam stable.
Proof : Let $v(x,t)$ is a solution to the inequality
$||v_{xx}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \epsilon, \forall\, x \in [0,a], \forall\, t \in [0,b]$.
Let $u(x,t)$ be the unique solution to the problem

$$\begin{aligned} u_{xx}(x,t) &= f(x,t,u(x,t),u_x(x,t)), \\ u(0,t) &= v(0,t), \forall\, t \in [0,b], \\ u(x,0) &= v(x,0), \forall\, x \in [0,a]. \end{aligned} \tag{3.1}$$

If $u(x,t)$ is a solution to (3.1) then $(u, u_x(x,t))$ satisfies the following system

$$\begin{aligned} u(x,t) &= v(0,t) + \int_0^x \int_0^y f(s,t,u(s,t),u_s(s,t))dsdy, \\ u_x(x,t) &= v_x(0,t) + \int_0^x f(s,t,u(s,t),u_s(s,t))ds. \end{aligned} \tag{3.2}$$

Then by using Theorem (2.6) it follows that,

$$||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy|| \leq \frac{\epsilon x^2}{2}, \tag{3.3}$$

and

$$||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds|| \leq \epsilon x. \quad (3.4)$$

Consider

$||v(x,t) - u(x,t)|| = ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,u(s,t),u_s(s,t))dsdy||.$

$= ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy$
$+ \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy$ - $\int_0^x \int_0^y f(s,t,u(s,t),u_s(s,t))dsdy||.$

$\leq ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy||$
$+|| \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy - \int_0^x \int_0^y f(s,t,u(s,t),u_s(s,t))dsdy||.$

$\leq ||v(x,t) - v(0,t) - \int_0^x \int_0^y f(s,t,v(s,t),v_s(s,t))dsdy||$
$+ \int_0^x \int_0^y ||f(s,t,v(s,t),v_s(s,t)) - f(s,t,u(s,v),u_s(s,t))||dsdy.$

$\leq \frac{\epsilon x^2}{2} + \int_0^x \int_0^y L_f \times max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}dsdy.$
(by using hypothesis (ii) and equation (3.3))

By using Lemma (2.5) we get,

$||v(x,t) - u(x,t)|| \leq \frac{\epsilon x^2}{2} \times exp\{\int_0^x \int_0^y L_f dsdy\}.$
$\leq \frac{\epsilon x^2}{2} \times exp\{L_f \frac{x^2}{2}\}.$
$\leq \frac{\epsilon a^2}{2} \times exp\{L_f \frac{a^2}{2}\}.$
$\leq \epsilon \times c_f^1,$ where $c_f^1 = \frac{a^2}{2} \times exp\{L_f \frac{a^2}{2}\}.$

Again consider,

$||v_x(x,t) - u_x(x,t)|| = ||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,u(s,t),u_s(s,t))ds||.$

$= ||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds$
$+ \int_0^x f(s,t,v(s,t),v_s(s,t))ds - \int_0^x f(s,t,u(s,t),u_s(s,t))ds||.$

$\leq ||v_x(x,t) - v_x(0,t) - \int_0^x f(s,t,v(s,t),v_s(s,t))ds||$
$+|| \int_0^x f(s,t,v(s,t),v_s(s,t))ds - \int_0^x f(s,t,u(s,t),u_s(s,t))ds||.$

$\leq \epsilon x + \int_0^x ||f(s,t,v(s,t),v_s(s,t)) - f(s,t,u(s,t),u_s(s,t))||ds.$
(by using theorem (2.6))

By using hypothesis (ii) we get,

$||v_x(x,t) - u_x(x,t)|| \leq \epsilon x$
+ $\int_0^x L_f \times max\{||v(s,t) - u(s,t)||, ||v_s(s,t) - u_s(s,t)||\}ds.$

By using Lemma (2.5) we get,

$||v_x(x,t) - u_x(x,t)|| \leq \epsilon x \times exp\{\int_0^x L_f ds\}.$
$\leq \epsilon x \times exp\{L_f\ x\}.$
$\leq \epsilon a \times exp\{L_f\ a\}.$
$\leq \epsilon \times c_f^2,$ where $c_f^2 = a \times exp\{L_f a\}.$

Hence equation (1.1) is Hyers-Ulam stable.

4 Hyers-Ulam stability for the second order non-linear partial differential equation :

In this section we prove the HU stability of second order non-linear ordinary differential equation (1.2) .

Theorem 4.1 : Assume that
i) $f \in C([0,a] \times [0,b] \times \mathbb{B}^2, \mathbb{B})$, where $a, b \in (0, \infty)$.
ii) There exists $L_f \in C^1([0,\infty) \times [0,\infty), \mathbb{R}_+)$ such that L_f is square inegrable and $||f(x,t,z_1,z_2) - f(x,t,t_1,t_2)|| \leq L_f \max_{i\in\{1,2\}} \{||z_i - t_i||\}, \forall\, x \in [0,a], \forall\, t \in [0,b]$ and $z_1, z_2, t_1, t_2 \in \mathbb{B}$.
iii) $\phi : (0,\infty) \times (0,\infty) \to (0,\infty)$ is an increasing function.
Then (1.2) is Hyers - Ulam stable.
Proof : Let $v(x,t)$ is a solution to the inequality
$||v_{xt}(x,t) - f(x,t,v(x,t),v_x(x,t))|| \leq \epsilon, \;\; \forall\, x \in [0,a], \forall\, t \in [0,b]$.
Let $u(x,t)$ be the unique solution to the problem

$$\begin{aligned} u_{xt}(x,t) &= f(x,t,u(x,t),u_x(x,t)),\\ u(0,t) &= v(0,t), \forall\, t \in [0,b],\\ u(x,0) &= v(x,0), \forall\, x \in [0,a]. \end{aligned} \tag{4.1}$$

If $u(x,t)$ is a solution to equation (4.1) then $(u, u_x(x,t), u_t(x,t))$ satisfies the following system

$$\begin{aligned} u(x,t) &= v(x,0) + v(0,t) - v(0,0) + \int_0^t \int_0^x f(s,z,u(s,z),u_s(s,z))dsdz,\\ u_t(x,t) &= v_t(0,t) + \int_0^x f(s,t,u(s,t),u_s(s,t))ds,\\ u_x(x,t) &= v_x(x,0) + \int_0^t f(x,z,u(x,z),u_x(x,z))dz. \end{aligned} \tag{4.2}$$

Then by using Theorem (2.7) it follows that,

$$\left|\left|v(x,t) - v(x,0) - v(0,t) + v(0,0) - \int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz\right|\right| \leq \epsilon xt, \tag{4.3}$$

and

$$||v_x(x,t) - v_x(x,0) - \int_0^t f(x,z,v(x,z),v_x(x,z))dz|| \leq \epsilon t. \tag{4.4}$$

Using equation (4.2) we get,

$$\begin{aligned} ||v(x,t) - u(x,t)|| &= ||v(x,t) - v(x,0) - v(0,t) + v(0,0)\\ &\quad - \textstyle\int_0^t \int_0^x f(s,z,u(s,z),u_s(s,z))dsdz||.\\ &= ||v(x,t) - v(x,0) - v(0,t) + v(0,0) - \textstyle\int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz\\ &\quad + \textstyle\int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz - \int_0^t \int_0^x f(s,z,u(s,z),u_s(s,z))dsdz||.\\ &\leq ||v(x,t) - v(x,0) - v(0,t) + v(0,0) - \textstyle\int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz||\\ &\quad + ||\textstyle\int_0^t \int_0^x f(s,z,v(s,z),v_s(s,z))dsdz - \int_0^t \int_0^x f(s,z,u(s,z),u_s(s,z))dsdz||. \end{aligned}$$

$$\leq ||v(x,t)-v(x,0)-v(0,t)+v(0,0)-\int_0^t\int_0^x f(s,z,v(s,z),v_s(s,z))dsdz||$$
$$+\int_0^t\int_0^x ||f(s,z,v(s,z),v_s(s,z))-f(s,z,u(s,z),u_s(s,z))||dsdz.$$

$$\leq \epsilon xt$$
$$+\int_0^t\int_0^x \Big\{L_f \times max\big[||v(s,z)-u(s,z)||,||v_s(s,z)-u_s(s,z)||\big]\Big\}dsdz,$$
(By using equation (4.3)).

By using Lemma (2.5) we get,
$||v(x,t)-u(x,t)|| \leq \epsilon xt \times exp\{\int_0^t\int_0^x L_f dsdz\}.$
$\leq \epsilon xt \times exp\{L_f\ xt\}.$
$\leq \epsilon ab \times exp\{L_f\ ab\}.$
$||v(x,t)-u(x,t)|| \leq \epsilon \times c_f^3,$ where $c_f^3 = ab \times exp\{L_f\ ab\}.$
Again by using equation (4.2) we get,
$||v_x(x,t)-u_x(x,t)|| = ||v_x(x,t)-v_x(x,0)-\int_0^t f(x,z,u(x,z),u_x(x,z))dz||.$

$$= ||v_x(x,t)-v_x(x,0)-\int_0^t f(x,z,v(x,z),v_x(x,z))dz$$
$$+\int_0^t f(x,z,v(x,z),v_x(x,z))dz-\int_0^t f(x,z,u(x,z),u_x(x,z))dz||.$$

$$\leq ||v_x(x,t)-v_x(x,0)-\int_0^t f(x,z,v(x,z),v_x(x,z))dz||$$
$$+||\int_0^t f(x,z,v(x,z),v_x(x,z))dz-\int_0^t f(x,z,u(x,z),u_x(x,z))dz||.$$

$$\leq ||v_x(x,t)-v_x(x,0)-\int_0^t f(x,z,v(x,z),v_x(x,z))dz||$$
$$+\int_0^t ||\big\{f(x,z,v(x,z),v_x(x,z))-f(x,z,u(x,z),u_x(x,z))\big\}dz||.$$

$$||v_x(x,t)-u_x(x,t)|| \leq \epsilon t$$
$$+\int_0^t \Big\{L_f \times max\big[||v(x,z)-u(x,z)||,||v_x(x,z)-u_x(x,z)||\big]\Big\}dz,$$
(By using equation (4.4) and hypothesis (ii)).

By using Lemma (2.5) we get,
$||v_x(x,t)-u_x(x,t)|| \leq \epsilon t \times exp\{\int_0^t L_f dz\}.$
$\leq \epsilon t \times exp\{L_f\ t\}.$
$\leq \epsilon b \times exp\{L_f\ b\}.$
$||v_x(x,t-u_x(x,t)|| \leq \epsilon \times c_f^4,$ where $c_f^4 = b \times exp\{L_f\ b\}.$
Hence equation (1.2) is Hyers-Ulam stable.

5 Conclusion

In this paper, we have proved the HU stability of second order non-linear ordinary and partial differential equations (1.1) and (1.2) respectively by emplyoing Gronwall type Lemma.

References

[1] Alsina C., Ger R., *on some inequalities and stability results related to the exponential function*, J. Inequal. Appl., **2**(1998), 373-380.

[2] Gordji M. E. , Cho Y. J., Ghaemi M. B., Alizadeh B. *stability of second order partial differential equations* , J. of Inequalities and Applications, **2011:81**(2011), 1-10.

[3] Gordji M. E., Kamiri T., Gharetapeh S. K. *Approximately n- Jordan homomorphisms on Banach Space algebras*, J. of Inequalities and Applications **2009**(2009),(Article ID 870843), 8 pages.

[4] Gordji M. E., Shams S., Ebadian A. and Ghaem M. B., *on the stability of j^* derivations*, J. Geom Phy. **60**(2010), 454-459.

[5] Hyers D. H., *on the stability of the linear functional equation*, proc. Natl., Acad. Science **27**(1941), 222-224.

[6] Jung S. M., *Hyers-Ulam stability of linear partial differential equation of first order* , Applied Mathematics Letters, **22**(2009),70 -74.

[7] Jung S. M., *Hyers-Ulam stability of linear differential equation of first order, II*, Applied Mathematics Letters, **19**(2006), 854 - 858.

[8] Jung S. M., *A fixed point approach to the stability of differential equations $y' = f(x, t)$* ,Bull Malays Sci. Soc. **33**(2010), 47- 56.

[9] Jung S. M., Min S., *On approximate Euler differential equations*. Abstr Ana **2009** (2009),(Article ID 537963),8 pages.

[10] LI Y., Shen Y., *Hyers-Ulam stability of linear differential equations of second order*, Applied Mathematics Letters, **23**(3)(2010), 306-309.

[11] Lunge, N., Cracium C., *Ulam- Hyers - Rassias stability of a hyperbolic partial differential equation* ,J. of Mathematical analysis, **2012**, Article ID 609754, 10 pages, doi : 10.5402/2012/609754.

[12] Pachpatte B. G., *Inequalities for Differential and Inegral equaions*, Academic Press, San Diego,Calif, USA, 1998.

[13] Rassias Th. M., *on the stability of linear mapping in Banach spaces* , proc. Amer. math. soc **72**(1978), 297-300.

[14] Rezaei H., Jung S. M., Rassias T. M., Laplace ransform and Hyers-Ulam stability of linear differential equations, *Journal of Mathematical Analysis and Applications*, **403**(2013), 244-251.

[15] Sonalkar V. P., Mohapatra A. N., Valaulikar Y. S., Hyers-Ulam- Rassias stability of linear partial differential equation, *Journal of Applied Science and Computations.*, **VI, Issue III**,(2019) (839-846), DOI:16.10089.JASC.2018V6I3.453459.150010361.

[16] Takahasi, S. E. , Miura, T., Miyajima, S., *Hyers-Ulam stability of Banach space valued linear differential equations $y' = \lambda y$*, Bull. of Korean Math. Soc., **39**(2002), 309-315.

[17] Ulam S. M., *A collection of Mathematical problems*, Interscience Tracts in Pure and Applied Mathematics, **8**, Interscience publication, New York, 1960.